THE ACTION OF A DIE ON AN ELASTIC LAYER OF FINITE THICKNESS

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Considered is the action of a rigid die on an elastic layer of thickness h resting on a rigid foundation. A method is presented for obtaining an approximate solution to the action of a die on an elastic layer, based on the known solution for the action of a die on an elastic half-space. The derived solution is valid for sufficiently large h and is given in the form of a series in powers of h^{-1}. Specific computing formulas are obtained for a die of elliptic plan form.

1. Statement of problem. Let a rigid die in the form of a cylindrical body with cross section Ω and foundation surface $z' = f(x', y')$ penetrate into an elastic layer resting on a rigid foundation (Fig. 1).

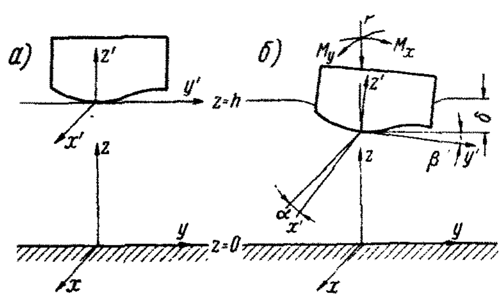

FIG. 1.

The die is subjected to a force P acting along the z-axis, and to moments M_x and M_y relative to the axes of the coordinates x and y.

Let us assume that the magnitude of the force and the moments is such that the region of contact for the die and the layer coincides with Ω. Furthermore, we assume that there are no frictional forces between the layer and the die, as well as between the layer and the foundation, and that outside the die the layer is not loaded. Under such assumptions the problem is reduced to the solution of the basic equations of the theory of elasticity subject to the following boundary conditions:

$$\tau_{zx} = 0, \quad \tau_{zy} = 0, \quad \sigma_z = 0 \text{ outside the region } \Omega \text{ for } z = h \tag{1.1}$$

$$w = -\delta(x, y) = -[\delta + \alpha x + \beta y - f(x, y)] \text{ in the region } \Omega \tag{1.2}$$

$$\tau_{zx} = 0, \quad \tau_{zy} = 0, \quad w = 0 \quad \text{for } z = 0 \tag{1.3}$$

Displacements and stresses vanish for $(x, y) \to \infty$. Here δ is the displacement of the die under the action of the force P, and α, β define the rotation angles of the die about the y- and z-axes, respectively, due to the moments M_y and M_x. Let us define the pressure between the layer and the die by

$$q(x, y) = -\sigma_z \quad \text{in the region } \Omega \text{ for } z = h \tag{1.4}$$

Note that in accordance with the physics of the problem $q(x, y) > 0$ and $\delta(x, y) > 0$ in Ω and also that the pressure $q(x, y)$ is associated by the known relations of statics with the force and the moments acting on the die:

$$P = \iint_\Omega q(x, y)\,dxdy, \quad M_x = \iint_\Omega yq(x, y)\,dxdy, \quad M_y = \iint_\Omega xq(x, y)\,dx\,dy \tag{1.5}$$

Assume at first that the pressure $q(x, y)$ between the die and the layer is known. Then we obtain the following relationships from [1] Chap. 3, Sect. 5, where the problem of compression of an elastic layer resting on a smooth rigid foundation and subject to a distributed normal loading $p(x, y)$ on its upper surface is considered:

$$\begin{aligned}
w &= \frac{h}{2\pi G}\iint_{-\infty}^{\infty} \frac{Q(\alpha, \beta)}{\gamma\Delta(\gamma h)}\, x_2(\gamma z)\, e^{i(\alpha x+\beta y)}\,d\alpha d\beta \qquad (\gamma = \sqrt{\alpha^2+\beta^2}) \\
\tau_{zx} &= -\frac{hi}{\pi}\iint_{-\infty}^{+\infty} \frac{Q(\alpha, \beta)\,\alpha}{\gamma\Delta(\gamma h)}\, x_3(\gamma z)\, e^{i(\alpha x+\beta y)}\,d\alpha d\beta \\
\tau_{zy} &= -\frac{hi}{\pi}\iint_{-\infty}^{\infty} \frac{Q(\alpha, \beta)\,\beta}{\gamma\Delta(\gamma h)}\, x_3(\gamma z)\, e^{i(\alpha x+\beta y)}\,d\alpha d\beta \qquad \Delta(\gamma h) = 2\gamma h + \operatorname{sh} 2\gamma h \\
\sigma_z &= -\frac{h}{\pi}\iint_{-\infty}^{\infty} \frac{Q(\alpha, \beta)\,\gamma}{\Delta(\gamma h)}\, x_4(\gamma z)\, e^{i(\alpha x+\beta y)}\,d\alpha d\beta
\end{aligned} \tag{1.6}$$

Here $Q(\alpha, \beta)$ is the Fourier transform of the function $p(x, y) = -[\sigma_z]_{z=h}$, i.e.

$$Q(\alpha,\ \beta) = -\frac{1}{2\pi}\iint\limits_{-\infty}^{\infty}[\sigma_z]_{z=h}e^{-i(\alpha x+\beta y)}\,dxdy \qquad (1.7)$$

$$x_2(\gamma z) = \frac{1}{h}\left[\gamma h\cosh\gamma h\sinh\gamma z - \gamma z\cosh\gamma z\sinh\gamma h + \frac{2(m-2)}{m}\sinh\gamma z\sinh\gamma h\right]$$
$$x_3(\gamma z) = \frac{1}{h}(\gamma h\cosh\gamma h\sinh\gamma z - \gamma z\cosh\gamma z\sinh\gamma h) \qquad (1.8)$$
$$x_4(\gamma z) = \cosh\gamma h\cosh\gamma z - \frac{z}{h}\sinh\gamma h\sinh\gamma z + \frac{\cosh\gamma z\sinh\gamma h}{\gamma h}$$

Formulas (1.6) are such that for any function $Q(\alpha,\ \beta)$, the first two boundary conditions (1.1) and the boundary conditions (1.3) are satisfied, as can be easily determined. Utilizing the third condition (1.1) and equality (1.4), we express (1.7) in the form

$$Q(\alpha,\ \beta) = \frac{1}{2\pi}\iint\limits_{\Omega} q(x,\ y)\,e^{-i(\alpha x+\beta y)}\,dxdy \qquad (1.9)$$

Finally, condition (1.2) will be satisfied if the function $Q(\alpha,\ \beta)$ is determined from the equation

$$-\frac{h}{2\pi G}\iint\limits_{-\infty}^{\infty}\frac{Q(\alpha,\ \beta)}{\gamma\Delta(\gamma h)}\,x_2(\gamma h)e^{i(\alpha x+\beta y)}d\alpha d\beta = -\delta(x,\ y) \text{ in the region } \Omega\left(\Delta = \frac{mG}{m-1}\right)$$

or

$$\iint\limits_{-\infty}^{\infty}\frac{\sinh^2\gamma h}{\gamma(2\gamma h+\sinh 2\gamma h)}\,Q(\alpha,\ \beta)\,e^{i(\alpha x+\beta y)}d\alpha d\beta = \pi\Delta\delta(x,\ y) \text{ in the region } \Omega \qquad (1.10)$$

Substituting $Q(\alpha,\ \beta)$ from (1.9) into (1.10), we reduce the problem to the determination of the function $q(x,\ y)$ from the integral equation of the first kind

$$\iint\limits_{\Omega} q(\xi,\ \eta)\,K(x-\xi,\ y-\eta,\ h)\,d\xi d\eta = 2\pi^2\Delta\delta(x,\ y) \qquad (1.11)$$

where

$$K(x-\xi,\ y-\eta,\ h) = \iint\limits_{-\infty}^{\infty}\frac{\sinh^2\gamma h}{\gamma(2\gamma h+\sinh 2\gamma h)}\,e^{i[\alpha(x-\xi)+\beta(y-\eta)]}\,d\alpha d\beta \qquad (1.12)$$

Noting that

$$K_\infty = \lim_{h\to\infty} K = \iint_{-\infty}^{\infty} e^{i[\alpha(x-\xi)+\beta(y-\eta)]}\frac{d\alpha d\beta}{2\gamma} = \frac{\pi}{\sqrt{(x-\xi)^2+(y-\eta)^2}} \qquad (1.13)$$

we shall express (1.11) in the form

$$\iint_\Omega \frac{q(\xi,\ \eta)\,d\xi d\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2}} = 2\pi\Delta\delta(x,\ y) - \frac{1}{\pi}\iint_\Omega q(\xi,\ \eta)(K-K_\infty)\,d\xi d\eta \qquad (1.14)$$

where

$$K - K_\infty = \iint_{-\infty}^{\infty} \frac{e^{-4\gamma h} - 2\gamma h e^{-2\gamma h} - e^{-2\gamma h}}{\gamma(4\gamma h e^{-2\gamma h} + 1 - e^{-4\gamma h})} e^{i[\alpha(x-\xi)+\beta(y-\eta)]}\,d\alpha d\beta \qquad (1.15)$$

Having determined $q(x,\ y)$ from Equation (1.14) we will find the relationships between the quantities P and δ, M_y and α, M_x and β from the relations (1.5), and the expression for the function $Q(\alpha,\ \beta)$ according to Formula (1.9); then by (1.6) and the analogous formulas for u, v, σ_x, σ_y and τ_{xy} in [1], we shall find the displacements and stresses in the layer.

2. Transformation of kernel $K - K_\infty$. Expanding the fraction under the double integral in (1.15) into a series of powers $l^{-2\gamma h} = t$ we obtain

$$\frac{t^2 - 2\gamma ht - t}{\gamma(4\gamma ht + 1 - t^2)} = \frac{1}{\gamma}(A_0 - A_1 t + A_2 t^2 - A_3 t^3 + \ldots)$$

$$t^2 - 2\gamma ht - t = (4\gamma ht + 1 - t^2)(A_0 - A_1 t + A_2 t^2 - A_3 t^3 + \ldots)$$

Equating the coefficients of equal powers of t we have

$$A_0 = 0, \qquad A_1 = 1 + 2\gamma h, \qquad A_2 = 1 + 4\gamma h + 8\gamma^2 h^2 \qquad (2.1)$$

$$A_n = A_{n-2} + 4\gamma h A_{n-1}, \qquad n = 3,\ 4, \ldots \qquad (2.2)$$

Let us denote the coefficients in A_k of $(\gamma h)^i$ by A_{ki} *then*

$$\frac{e^{-4\gamma h} - 2\gamma h e^{-2\gamma h} - e^{-2\gamma h}}{4\gamma h e^{-2\gamma h} + 1 - e^{-4\gamma h}} = \frac{1}{\gamma}\sum_{k=1}^{\infty}(-1)^k e^{-2\gamma hk}\sum_{i=0}^{k} A_{ki}(\gamma h)^i \qquad (2.3)$$

The coefficients A_{ki} are determined by the relations

$$\begin{aligned}
&A_{00} = 0, \qquad A_{2r+2,\,0} = 1 \\
&A_{2r+1,\,2s} = \frac{(r+s)!}{(2s)!\,(r-s)!}\,2^{4s} \\
&A_{2r+1,\,2s+1} = \frac{(2r+1)\,(r+s)!}{(2s+1)!\,(r-s)!}\,2^{4s+1} \qquad \begin{pmatrix} s = 0,\ 1,\ 2, \ldots, r \\ r = 0,\ 1,\ 2, \ldots \end{pmatrix} \qquad (2.4)\\
&A_{2r+2,\,2s+1} = \frac{(r+s+1)!}{(2s+1)!\,(r-s)!}\,2^{4s+2} \\
&A_{2r+2,\,2s+2} = \frac{(2r+2)\,(r+s+1)!}{(2s+2)!\,(r-s)!}\,2^{4s+3}
\end{aligned}$$

Utilizing the expansion

$$e^{i[\alpha(x-\xi)+\beta(y-\eta)]} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{i^{m+n}\alpha^m\beta^n\,(x-\xi)^m\,(y-\eta)^n}{m!\,n!}$$

and Formula (2.3), we will have

$$K - K_\infty = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_{mn}\,(x-\xi)^m\,(y-\eta)^n \qquad (2.5)$$

where

$$C_{mn} = \frac{i^{m+n}}{m!\,n!}\sum_{k=1}^{\infty}(-1)^k \sum_{i=0}^{k} A_{ki}\,h^i \int_0^{2\pi}\cos^m\varphi\,\sin^n\varphi\,d\varphi \int_0^{\infty}\gamma^{m+n+i}e^{-2k\gamma h}d\gamma \qquad (2.6)$$

Evaluating the integrals in (2.6) we will find

$$C_{2m+1,\,2n} = C_{2m,\,2n+1} = C_{2m+1,\,2n+1} = 0, \quad C_{2m,\,2n} = \frac{2\pi}{h^{2m+2n+1}}\,\Gamma_{mn} \qquad (2.7)$$

Here (2.8)

$$\Gamma_{mn} = \frac{(-1)^{m+n+1}(2m+2n)!}{(m+n)!\,m!\,n!\,2^{4m+4n+1}}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{2m+2n+1}}\sum_{i=0}^{k}A_{ki}\frac{(2m+2n+1)\ldots(2m+2n+i)}{(2k)^i}$$

Substituting expressions A_{ki} from Formulas (2.4) and denoting $m+n=p$, we reduce $\Gamma_{mn}m!n! = \Gamma_p$ to

$$\Gamma_p = \frac{(-1)^p}{p!\,2^{6p+2}}\sum_{r=0}^{\infty}\left\{\sum_{s=0}^{r}\left[-\left(2+\frac{p}{s+{}^1\!/_2}\right)\frac{(r+s)!\,(2p+2s)!}{(2s)!\,(r-s)!\,(r+{}^1\!/_2)^{2p+2s+1}} + \right.\right.$$

$$\left.\left. + \left(2+\frac{p}{s+1}\right)\frac{(r+s+1)!\,(2p+2s+1)!}{(2s+1)!\,(r-s)!\,(r+1)^{2p+2s+2}}\right] + \frac{(2p)!}{(r+1)^{2p+1}}\right\} \qquad (2.9)$$

Computations yield

$$\Gamma_{00} = -0.5838 \pm 0.0001, \qquad \Gamma_{10} = \Gamma_{01} = 0.1977 \pm 0.0001 \quad (2.10)$$

Substituting C_{mn} from Formulas (2.7) into (2.5), we have

$$K - K_\infty = 2\pi \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(x-\xi)^{2m} (y-\eta)^{2n}}{h^{2m+2n+1}} \Gamma_{mn} \qquad (2.11)$$

3. Solution of Equation (1.14). Substituting the expression (2.11) into (1.14) we obtain

$$\int_\Omega \frac{q(\xi, \eta)\, d\xi d\eta}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} =$$

$$= 2\pi\Delta\delta(x, y) - 2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\Gamma_{mn}}{h^{2m+2n+1}} \int_\Omega q(\xi, \eta)(x-\xi)^{2m} (y-\eta)^{2n}\, d\xi d\eta \qquad (3.1)$$

It is easy to show that the series (2.11) converges to $K - K_\infty$ for any $x - \xi$, $y - \eta$ and for $h \geqslant d/\sqrt{2}$.

The convergence of the series is not deteriorated by partial integration, therefore Equation (3.1) is valid at least for

$$h \geqslant \frac{d}{\sqrt{2}} \qquad (d = \max \sqrt{(x-\xi)^2 + (y-\eta)^2}, \quad (x, y) \in \Omega\, u\, (\xi, \eta) \in \Omega)$$

We shall search for $q(\xi, \eta)$ in the form

$$q(\xi, \eta) = \sum_{i=0}^{\infty} q_i(\xi, \eta) \frac{1}{h^i} \qquad (3.2)$$

Substituting $q(\xi, \eta)$ from (3.2) into (3.1) and equating the terms of like powers of h^{-1}, we obtain the integral equations

$$\int_\Omega \frac{q_0(\xi, \eta)\, d\xi d\eta}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} = 2\pi\Delta\delta(x, y) \qquad (3.3)$$

$$\int_\Omega \frac{q_1(\xi, \eta)\, d\xi d\eta}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} = -2\Gamma_{00} \int_\Omega q_0(\xi, \eta)\, d\xi, d\eta \qquad (3.4)$$

$$\int_\Omega \frac{q_2(\xi, \eta)\, d\xi d\eta}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} = -2\Gamma_{00} \int_\Omega q_1(\xi, \eta)\, d\xi d\eta \qquad (3.5)$$

$$\int \frac{q_3(\xi, \eta)\, d\xi d\eta}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} = -2 \int_\Omega [(x-\xi)^2 \Gamma_{10} q_0 + \Gamma_{00} q_2 + (y-\eta)^2 \Gamma_{01} q_0]\, d\xi d\eta \qquad (3.6)$$

$$\int_{\Omega} \frac{q_4(\xi, \eta)\, d\xi d\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2}} = -2 \int_{\Omega} [(x-\xi)^2\Gamma_{10}q_1 + \Gamma_{00}q_3 + (y-\eta)^2\Gamma_{01}q_1]\, d\xi d\eta \quad (3.7)$$

Let us assume now that we know how to solve the problem for the die on an elastic half-space. This means that we can obtain $q_0(\xi, \eta)$ from Equation (3.3). Substituting into the right-hand side of Equation (3.4), we will obtain an integral equation of the type (3.3) which consequently yields $q_1(\xi, \eta)$ etc.

Note that the kernel $K(x - \xi, y - \eta, h)$ in Equation (1.11) is symmetric with respect to the variables x, y and ξ, η. We can then generalize Equations (2) given in [3].

Namely, if the solution is known for the action of a flat die with the region of contact Ω on an elastic layer of finite depth, i.e. we know the solution

$$q_{\text{pl}}(x, y) = \delta q_\delta(x, y) + \alpha q_\alpha(x, y) + \beta q_\beta(x, y) \quad (3.8)$$

of the equation

$$\int_{\Omega} q_{\text{pl}}(\xi, \eta)\, K(x-\xi, y-\eta, h)\, d\xi d\eta = 2\pi^2 \Delta(\delta + \alpha x + \beta y) \quad (3.9)$$

then the force and the moments acting on the die with an arbitrary foundation and the same region of contact Ω are expressed by formulas

$$P = \int_{\Omega} q_\delta(x, y)\, \delta(x, y)\, dxdy, \qquad M_y = \int_{\Omega} q_\alpha(x, y)\, \delta(x, y)\, dxdy$$

$$M_x = \int_{\Omega} q_\beta(x, y)\, \delta(x, y)\, dxdy \quad (3.10)$$

4. Solution of the problem for a flat elliptic die. Let us solve the problem of the action of an elliptic flat die on a layer of finite depth by means of the method presented above.*

First of all, we give the solution for the elliptic die on an elastic half-space obtained by Galin (see [2], Chap. 2, Sect. 8) and at the same time correct an error which occurred in this work.

Following Galin, we will write the expression for the potential of a simple layer $W(x, y, x)$, located on the surface of the ellipsoid $\rho = \kappa$:

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* The solution by the following method can also be obtained for a non-plane elliptic die.

$$W = \begin{cases} W_1(x, y, z) = \sum_{k=0}^{n} \sum_{m=1}^{2k+1} A_{km} E_k{}^m(\rho) E_k{}^m(\mu) E_k{}^m(\nu) & \begin{pmatrix}\text{Harmonic} \\ \text{function inside} \\ \text{ellipsoid } \rho = \kappa\end{pmatrix} \\ W_2(x, y, z) = \sum_{k=0}^{n} \sum_{m=1}^{2k+1} A_{km} \frac{E_k{}^m(\varkappa)}{F_k{}^m(\varkappa)} F_k{}^m(\rho) E_k{}^m(\mu) E_k{}^m(\nu) & \begin{pmatrix}\text{Harmonic func-} \\ \text{tion outside} \\ \text{ellipsoid } \rho = \kappa \\ \text{vanishing at } \infty\end{pmatrix} \end{cases} \tag{4.1}$$

At the surface of the ellipsoid

$$W(x, y, z)|_{\rho=\varkappa} = W_1|_{\rho=\varkappa} = W_2|_{\rho=\varkappa} = \sum_{k=0}^{n} \sum_{m=1}^{2k+1} A_{km} E_k{}^m(\varkappa) E_k{}^m(\mu) E_k{}^m(\nu) = $$
$$= 2\pi \Delta Q_n(x, y, z) \tag{4.2}$$

where $Q(x, y, z)$ is a polynomial of order n, even on z; ρ, μ, ν are ellipsoidal coordinates connected with right-angle relations

$$\begin{gathered} l^2 x^2 = a^2 \rho^2 \mu^2 \nu^2 \\ (1 - l^2) l^2 y^2 = a^2 (\rho^2 - l^2)(\mu^2 - l^2)(l^2 - \nu^2) \\ (1 - l^2) z^2 = a^2 (\rho^2 - 1)(1 - \mu^2)(1 - \nu^2) \\ (0 \leqslant \nu^2 \leqslant l^2,\ l^2 \leqslant \mu \leqslant 1,\ 1 \leqslant \rho^2 \leqslant \infty) \end{gathered} \tag{4.3}$$

Here $E_k{}^m(\rho)$ and $F_k{}^{(m)}(\rho)$ are Lamé functions of the first and second kind,* whereby

$$F_k{}^m(\rho) = E_k{}^m(\rho)\psi_k{}^m(\rho), \qquad \psi_k{}^m(\rho) = \int_\rho^\infty \frac{d\rho}{[E_k{}^m(\rho)]^2 \sqrt{(\rho^2 - l^2)(\rho^2 - 1)}} \to 0 \text{ for } \rho \to \infty \tag{4.4}$$

Now let us write the expression for the density of the potential for the simple layer located on the surface of the ellipsoid $\rho = \kappa$:

$$q(x, y, z) = -\frac{1}{4\pi}\left[\left(\frac{\partial W_2}{\partial \rho} - \frac{\partial W_1}{\partial \rho}\right)\frac{\partial \rho}{\partial n}\right]_{\rho=\varkappa} \tag{4.5}$$

Substituting expressions W_1 and W_2 from (4.1) into (4.5) and transforming we obtain

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* For the theory of Lamé functions see [4] Chap. 23; [2] Chap. 2, Sects. 2, 8, 9; [1] Chap. 5, Sect. 8; [5] Chap. 6, Sects. 184-186.

$$q(x,y,z) = -\frac{1}{4\pi}\left(\frac{\partial\rho}{\partial n}\right)_{\rho=\varkappa}\sum_{k=0}^{n}\sum_{m=1}^{2k+1} A_{km}\left[\frac{d}{d\rho}\left(\frac{\psi_k{}^m(\rho)}{\psi_k{}^m(\varkappa)}E_k{}^m(\rho) - E_k{}^m(\rho)\right)\right]_{\rho=\varkappa} \times$$
$$\times E_k{}^m(\mu)E_k{}^m(\nu) = -\frac{1}{4\pi}\left(\frac{\partial\rho}{\partial n}\right)_{\rho=\varkappa}\sum_{k=0}^{n}\sum_{m=1}^{2k+1} A_{km}E_k{}^m(\varkappa)E_k{}^m(\mu)E_k{}^m(\nu)\frac{[\psi_k^m(\varkappa)]'}{\psi_k^m(\varkappa)} \tag{4.6}$$

Substituting

$$[\psi_k{}^m(\varkappa)]' = -\frac{1}{[E_k{}^m(\varkappa)]^2\sqrt{(\varkappa^2-l^2)(\varkappa^2-1)}}, \qquad \frac{\partial n}{\partial\rho} = H_\rho = a\sqrt{\frac{(\rho^2-\mu^2)(\rho^2-\nu^2)}{(\rho^2-l^2)(\rho^2-1)}} \tag{4.7}$$

we reduce (4.6) to

$$q(x,y,z) = \frac{1}{4\pi a\sqrt{(\varkappa^2-\mu^2)(\varkappa^2-\nu^2)}}\sum_{k=0}^{n}\sum_{m=1}^{2k+1} A_{kl}\frac{E_k{}^m(\mu)E_k{}^m(\nu)}{E_k{}^m(\varkappa)\psi_k{}^m(\varkappa)} \tag{4.8}$$

Let $\varkappa$ go to unity, then $z \to 0$ and the ellipsoid $\rho = \varkappa$ degenerates into an elliptic disc on the surface $z = 0$, the semiaxes of which are a and $b = a\sqrt{1-l^2}$. Thus, at the surface of the elliptic disc the potential W becomes

$$W(x,y,0) = \sum_{k=0}^{n}\sum_{m=1}^{2k+1} A_{km}E_k{}^m(1)E_k{}^m(\mu)E_k{}^m(\nu) = 2\pi\Delta Q_n(x,y,0) = 2\pi\Delta P_n(x,y) \tag{4.9}$$

Note that

$$\sqrt{(1-\mu^2)(1-\nu^2)} = \sqrt{1-l^2}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}} \tag{4.10}$$

Letting, now, $\varkappa$ go to unity in Expression (4.8) and multiplying the result by two, we will find the density of the potential of a simple layer located on the elliptic disc:

$$q(x,y) = \frac{1}{2\pi b}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{-\frac{1}{2}}\sum_{k=0}^{n}\sum_{m=1}^{2k+1} A_{km}\frac{E_k^m(1)E_k{}^m(\mu)E_k{}^m(\nu)}{[E_k{}^m(1)]^2\psi_k{}^m(1)} \tag{4.11}$$

For the particular case

$$P_2(x,y) = \delta + \alpha x + \beta y - Ax^2 - By^2$$

and using the theory of Lamé functions we can obtain from (4.11) and (4.9)

$$q(x, y) = \frac{\Delta}{b}\left(1 - \frac{x^2}{a^4} - \frac{y^2}{b^2}\right)^{-\frac{1}{2}} \left\{\frac{1}{K(l)}\left[\delta - \frac{1}{3}(Aa^2 + Bb^2)\right] + \right. \tag{4.12}$$

$$+ \frac{\alpha x l^2}{K(l) - E(l)} + \frac{\beta y l^2}{E(l) - (1 - l^2)K(l)} -$$

$$\left. - \frac{a^2}{l^2\tau} \sum_{=1}^{2} \frac{(-1)^i [A\sigma_{3-i} - B(\sigma_{3-i} - l^2)]\, \sigma_i^2 (\sigma_i - l^2)^2}{(\sigma_i - 1)[E(l) - (1 - \sigma_i) K(l)]} \left(\frac{x^2}{a^2\sigma_i} + \frac{y^2}{a^2(\sigma_i - l^2)} - 1\right)\right\}$$

$$\left(\sigma_{1,2} = \frac{1 + l^2}{3} \pm \tau;\ \tau = \frac{1}{3}\sqrt{1 - l^2 + l^4}\right)$$

Thus, Formula (4.12) yields the solution of the equation

$$\int_\Omega \frac{q(\xi, \eta)\, d\xi\, d\eta}{\sqrt{(x - \xi)^2 + (y - \eta)^2}} = 2\pi\Delta\delta(x, y)$$

to which is reduced the problem of the action of a die on an elastic half-space for the case when $\delta(x, y) = P_2(x, y)$ and Ω is an ellipse with semiaxes a and b.

We pass now to the solution of the action of a flat elliptic die on an elastic layer of finite thickness h. For a flat die $f(x, y) = 0$ and $\delta(x, y) = \delta + \alpha x + \beta y$. Therefore, assuming $A = B = 0$ in Formula (4.12), we obtain the solution of Equation (3.3):

$$q_0(x, y) = \frac{\Delta}{b}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-\frac{1}{2}} \left[\frac{\delta}{K(l)} + \frac{\alpha x l^2}{K(l) - E(l)} + \frac{\beta y l^2}{E(l) - (1 - l^2)K(l)}\right] \tag{4.13}$$

Substituting the expression obtained for q_0 into the right-hand side of Equation (3.4) we will obtain

$$\int_\Omega \frac{q_1(\xi, \eta)\, d\xi\, d\eta}{\sqrt{(x - \xi)^2 + (y - \eta)^2}} = 2\pi\Delta\left(-\frac{2\Gamma_{00} a\delta}{K(l)}\right)$$

and from this, using (4.12) and assuming $A = B = \alpha = \beta = 0$, we will find

$$q_1(x, y) = -\frac{2\Gamma_{00}\, a\delta\Delta}{b\,[K(l)]^2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-\frac{1}{2}} \tag{4.14}$$

Analogously, from Equation (3.5), we determine

$$q_2(x, y) = \frac{4\Gamma_{00}^2 a^2\delta\Delta}{b\,[K(l)]^3}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-\frac{1}{2}} \tag{4.15}$$

Substituting now the expressions for q_0, q_1, q_2 into (3.6) and rearranging we obtain

$$\int_{\Omega} \frac{q_3(\xi, \eta)\, d\xi\, d\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2}} = 2\pi\Delta\, [\delta_3 + \alpha_3 x + \beta_3 y - A_3 (x^2+y^2)]$$

Here

$$\delta_3 = -\frac{2^3\Gamma_{00}{}^3 a^3 \delta}{[K(l)]^3} - \frac{2\Gamma_{10}\delta a}{3K(l)}(a^2+b^2), \qquad \alpha_3 = \frac{4\Gamma_{10}\alpha l^2 a^3}{3[K(l)-E(l)]}$$

$$\beta_3 = \frac{4\Gamma_{10}\beta l^2 a b^2}{3\,[E(l) - (1-l^2)K(l)]}, \qquad A_3 = \frac{2\Gamma_{10}\delta a}{K(l)}$$

Using Formula (4.12) and in it letting $A = B$ we find

$$q_3(x, y) = -\frac{2a^3\Delta}{b}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{-\frac{1}{2}}\left[\frac{2^2\Gamma_{00}{}^3\delta}{[K(l)]^4} + \frac{2\Gamma_{10}\delta(2-l^2)}{3\,[K(l)]^2} - \frac{2\Gamma_{10}\alpha l^4}{3[K(l)-E(l)]^2}x - \right.$$
$$- \frac{2\Gamma_{10}\beta l^4(1-l^2)}{3\,[E(l)-(1-l^2)K(l)]^2}\, y + \frac{\Gamma_{10}\delta}{\tau K(l)}\sum_{i=1}^{2}\frac{(-1)^i\sigma_i^2(\sigma_i - l^2)^2}{(\sigma_i-1)[E(l)-(1-\sigma_i)K(l)]}\times$$
$$\left.\times\left(\frac{x^2}{a^2\sigma_i} + \frac{y^2}{a^2(\sigma_i - l^2)} - 1\right)\right] \qquad (4.16)$$

Analogously, from Equation (3.7) we determine

$$q_4(x, y) = \frac{4\Delta\delta\Gamma_{00}a^4}{b\,[K(l)]^2}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{-\frac{1}{2}}\left[\frac{4\Gamma_{00}{}^3}{[K(l)]^3} + \frac{4\Gamma_{10}(2-l^2)}{3K(l)} + \right.$$
$$\left. + \frac{\Gamma_{10}}{\tau}\sum_{i=1}^{2}\frac{(-1)^i\sigma_i^2(\sigma_i-l^2)^2}{(\sigma_i-1)\,[E(l)-(1-\sigma_i)K(l)]}\left(\frac{x^2}{a^2\sigma_i}+\frac{y^2}{a^2(\sigma_i-l^2)}-1\right)\right] \text{ (etc.) } (4.17)$$

And so we have found the first four terms of series (3.2).

The obtained approximate expression for $q(x, y)$ will have the form

$$q(x, y) = \delta q_\delta(x, y) + \alpha q_\alpha(x, y) + \beta q_\beta(x, y)$$

where

$$q_\delta(x, y) = \frac{\Delta}{bK(l)}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{-\frac{1}{2}}[R(h) + S(h, x^2, y^2)] \qquad (4.18)$$

$$q_\alpha(x, y) = \frac{\Delta l^2 x}{b[K(l)-E(l)]}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{-\frac{1}{2}} T_\alpha(h) \qquad (4.19)$$

$$q_\beta(x, y) = \frac{\Delta l^2 y}{b\,[E(l)-(1-l^2)K(l)]}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{-\frac{1}{2}} T_\beta(h) \qquad (4.20)$$

$$R(h) = 1 - \frac{2\Gamma_{00}a}{hK(l)} + \left(\frac{2\Gamma_{00}a}{hK(l)}\right)^2 - \left(\frac{2\Gamma_{00}a}{hK(l)}\right)^3 + \left(\frac{2\Gamma_{00}a}{hK(l)}\right)^4 - \\ - \frac{2^2a^3\Gamma_{10}(2-l^2)}{3h^3K(l)}\left[1 - \frac{2^2\Gamma_{00}a}{hK(l)}\right] + O\left(\frac{1}{h^4}\right) \tag{4.21}$$

$$S(h, x^2, y^2) = -\frac{2a^3\Gamma_{10}}{h^3\tau}\sum_{i=1}^{2}\frac{(-1)^i\sigma_i^2(\sigma_i - l^2)^2}{(\sigma_i - 1)[E(l) - (1-\sigma_i)K(l)]}\left(\frac{x^2}{a^2\sigma_i} + \right. \\ \left. + \frac{y^2}{a^2(\sigma_i - l^2)} - 1\right)\left[1 - \frac{2a\Gamma_{00}}{hK(l)}\right] + O\left(\frac{1}{h^4}\right)$$

$$T_\alpha(h) = 1 + \frac{4a^3\Gamma_{10}l^2}{3h^3[K(l) - E(l)]} + O\left(\frac{1}{h^4}\right) \\ T_\beta(h) = 1 + \frac{4a^3\Gamma_{10}l^2(1-l^2)}{3h^3[E(l) - (1-l^2)K(l)]} + O\left(\frac{1}{h^4}\right) \tag{4.22}$$

We will determine the relations between P and δ, M_y and α, M_x and β according to (1.5):

$$\frac{P}{\delta} = \int_\Omega q_\delta(x, y)\,dxdy = \frac{2\pi a\Delta}{K(l)}R(h) \\ \frac{M_y}{\alpha} = \int_\Omega xq_\alpha(x, y)\,dxdy = \frac{2\pi a^3l^2\Delta}{3[K(l) - E(l)]}T_\alpha(h) \\ \frac{M_x}{\beta} = \int_\Omega yq_\beta(x, y)\,dxdy = \frac{2\pi a^3l^2(1-l^2)\Delta}{3[E(l) - (1-l^2)K(l)]}T_\beta(h) \tag{4.23}$$

Rewriting, now, Formulas (3.10), and substituting q_δ, q_α and q_β from (4.18) and (4.19)

$$P = \frac{\Delta}{bK(l)}\int_\Omega [R(h) + S(h, x^2, y^2)]\,\delta(x, y)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-\frac{1}{2}}dxdy \\ M_y = \frac{\Delta l^2}{b[K(l) - E(l)]}T_\alpha(h)\int_\Omega x\delta(x, y)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-\frac{1}{2}}dxdy \\ M_x = \frac{\Delta l^2}{b[E(l) - (1-l^2)K(l)]}T_\beta(h)\int_\Omega y\delta(x, y)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{-\frac{1}{2}}dxdy \tag{4.24}$$

Formulas (4.24) permit the determination of the force and moments acting on any non-plane elliptic die. For $h = \infty$ we have $R = T_\alpha = T_\beta = 1$, $S = 0$, and Formulas (4.24) become the known formulas obtained by Galin ([21] Chap. 2, Sect. 9).

5. Derivation of computing formulas. Let us express $q(x, y)$ in the form

$$q(x, y) = \Delta\left(a^2 - x^2 - \frac{1}{1-l^2}y^2\right)^{-\frac{1}{2}}\left\{\delta\left(A + B\frac{a}{h} + C\frac{a^2}{h^2} + D\frac{a^3}{h^3} + E\frac{ax^2}{h^3} + \right.\right.$$
$$\left. + F\frac{ay^2}{h^3} + G\frac{a^4}{h^4} + H\frac{a^2x^2}{h^4} + I\frac{a^2y^2}{h^4}\right) +$$
$$\left. + \alpha x\left(K + L\frac{a^3}{h^3}\right) + \beta y\left(M + N\frac{a^3}{h^3}\right) + O\left(\frac{1}{h^4}\right)\right\} \quad (5.1)$$

whereby

$$\frac{R(h) + S(h, x^2, y^2)}{\sqrt{1-l^2}K(l)} = A + B\frac{a}{h} + C\frac{a^2}{h^2} + D\frac{a^3}{h^3} + E\frac{ax^2}{h^3} + F\frac{ay^2}{h^3} + G\frac{a^4}{h^4} +$$
$$+ H\frac{a^2x^2}{h^4} + I\frac{a^2y^2}{h^4} \quad (5.2)$$

$$\frac{R(h)}{\sqrt{1-l^2}K(l)} = A + B\frac{a}{h} + C\frac{a^2}{h^2} + D\frac{a^3}{h^3} + G\frac{a^4}{h^4} + E\frac{a^3}{3h^2} + F\frac{a^3(1-l^2)}{3h^3} +$$
$$+ H\frac{a^4}{3h^4} + I\frac{a^4(1-l^2)}{3h^4} \quad (5.3)$$

$$\frac{l^2T_\alpha(h)}{\sqrt{1-l^2}[K(l) - E(l)]} = K + L\frac{a^3}{h^3}, \quad \frac{l^2T_\beta(h)}{\sqrt{1-l^2}[E(l) - (1-l^2)K(l)]} = M + N\frac{a^3}{h^3} \quad (5.4)$$

Here the coefficients $A, B, \ldots, N$ depend only on the eccentricity l of the elliptic region of contact,

The table below gives the numerical values of these eccentricities l in the region $0 < l^2 < 0.99$. The intermediate values of eccentricities l can be obtained by interpolation. After determination of the coefficients for a given value of l, the computation of stress $q(x, y)$ according to (5.1) for various values of x and y in the region $-a < x < a$, $-b < y < b$ is not difficult.

The last column of the table gives the smallest values of the ratio h/a for which Formula (5.1) is still valid. These smallest values are determined for the case $\alpha = \beta = 0$ as follows. Let us introduce the notation

$$q^n(x, y) = q_0(x, y) + \frac{q_1(x, y)}{h} + \ldots + \frac{q_n(x, y)}{h^n} \quad (5.5)$$

and determine the quantities

$$\lambda_{\substack{x=0\\y=0}} = \frac{|q^4(0, 0) - q^3(0, 0)|}{q^3(0, 0)}100\% = \frac{|G|\frac{a^4}{h^4}100\%}{A + B\frac{a}{h} + C\frac{a^2}{h^2} + D\frac{a^3}{h^3}} \quad (5.6)$$

$$\lambda_{\substack{x=0\\y=b}} = \lim_{y\to b}\frac{|q^4(0, y) - q^3(0, y)|}{q^3(0, y)}100\% = \frac{\frac{a^4}{h^4}|G + (1-l^2)I|100\%}{A + B\frac{a}{h} + C\frac{a^2}{h^2} + D\frac{a^3}{h^3} + F\frac{a^3(1-l^2)}{h^3}} \quad (5.7)$$

$$\lambda_{\substack{x=a\\y=0}} = \lim_{x\to a} \frac{|q^4(x,0) - q^3(x,0)|}{q^3(x,0)} 100\% = \frac{\frac{a^4}{h^4}|G+H|100\%}{A + B\frac{a}{h} + C\frac{a^2}{h^2} + D\frac{a^3}{h^3} + E\frac{a^3}{h^3}} \quad (5.8)$$

It is easily seen from the table that

$$\lambda_{\substack{x=a\\y=0}} \geqslant \lambda_{\substack{x=0\\y=b}}, \qquad \lambda_{\substack{x=a\\y=0}} > \lambda_{\substack{x=0\\y=0}} \quad (5.9)$$

TABLE

l^2	$\frac{1}{1-l^2}$	$\sqrt{1-l^2}$	A	B	C	D	E	F
0	1.0	1.0	0.6366	0.4732	0.3517	0.4751	—0.6410	—0.6410
0.1	1.1111	0.9487	0.6537	0.4734	0.3428	0.4508	—0.6246	—0.6583
0.2	1.25	0.8944	0.6737	0.4739	0.3334	0.4254	—0.6072	—0.6788
0.3	1.4286	0.8367	0.6974	0.4751	0.3237	0.3985	—0.5888	—0.7034
0.4	1.6667	0.7746	0.7263	0.4771	0.3134	0.3700	—0.5691	—0.7339
0.5	2.0	0.7071	0.7628	0.4803	0.3025	0.3393	—0.5479	—0.7728
0.6	2.5	0.6325	0.8110	0.4857	0.2909	0.3057	—0.5249	—0.8250
0.7	3.3333	0.5477	0.8797	0.4949	0.2784	0.2680	—0.4997	—0.9004
0.8	5.0	0.4472	0.9906	0.5124	0.2651	0.2239	—0.4720	—1.0236
0.85	6.6667	0.3873	1.0808	0.5282	0.2582	0.1978	—0.4576	—1.1248
0.9	10.0	0.3162	1.2266	0.5555	0.2516	0.1669	—0.4438	—1.2895
0.95	20.0	0.2236	1.5377	0.6173	0.2478	0.1261	—0.4367	—1.6429
0.99	100.0	0.1	2.7059	0.8549	0.2701	0.06478	—0.4934	—2.9772

l^2	G	H	I	K	L	M	N	$\frac{h}{a}$
0	0.1943	—0.4765	—0.4765	1.2732	0.4273	1.2732	0.4273	1.52
0.1	0.1794	—0.4523	—0.4767	1.2905	0.4164	1.3249	0.3951	1.50
0.2	0.1638	—0.4272	—0.4776	1.3108	0.4051	1.3860	0.3623	1.48
0.3	0.1473	—0.4011	—0.4792	1.3353	0.3932	1.4597	0.3290	1.46
0.4	0.1298	—0.3738	—0.4821	1.3657	0.3808	1.5513	0.2948	1.44
0.5	0.1112	—0.3450	—0.4867	1.4046	0.3677	1.6693	0.2597	1.41
0.6	0.09112	—0.3143	—0.4941	1.4570	0.3539	1.8293	0.2232	1.38
0.7	0.06905	—0.2811	—0.5065	1.5330	0.3393	2.0644	0.1846	1.34
0.8	0.04400	—0.2442	—0.5295	1.6583	0.3242	2.4604	0.1427	1.29
0.85	0.02965	—0.2236	—0.5497	1.7619	0.3169	2.7956	0.1197	1.26
0.9	0.01311	—0.2010	—0.5840	1.9317	0.3111	3.3603	0.09412	1.22
0.95	—0.008116	—0.1753	—0.6596	2.2992	0.3116	4.6429	0.06353	1.16
0.99	—0.04112	—0.1559	—0.9406	3.6945	0.3598	10.1120	0.02695	1.04

Now, the smallest permissible value of the ratio h/a is determined from the condition

$$\lambda_{\substack{x=a \\ y=0}} \leqslant 5\% \tag{5.10}$$

i.e. in such a way that the transfer from $q^3(x, y)$ to $q^4(x, y)$ would not alter the quantity $q^3(x, y)$ by more than 5% for all $(x, y) \in \Omega$.

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